Testing Different Formulations of MOND Using LISA Pathfinder

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Previously it has been shown that MOdified Newtonian Dynamics (MOND) can be tested near the saddle points of the Newtonian gravitational potential using the forthcoming LISA Pathfinder (LPF) mission. These analysis focused on one particular formulation of the MONDian theory, which centered around a particular modified Poisson equation. We show that, in addition to the well known AQUAL formulation, another possibility exists in the form of a driven Poisson equation for the MONDian field. We consider similar quantitative and qualitative analysis for this formulation and also investigate typical Signal to Noise Ratios (SNR) resulting from these theories for a LPF test. We demonstrate that for a typical 50 km saddle flyby, the SNR is amplified by 25% between the two formulations. We also suggest that the SNR remains enhanced for impact parameters of 1000km or larger. Finally we show that constraints from a negative result remain as good, but no better between different formulations.

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I. INTRODUCTION

In the past few decades, attempts to reconcile anomalous observations of galactic dynamics without resorting to dark matter have been one of the driving forces behind modified gravity theories. These theories have centred around using a prescription for a modified force law [1], which suggests the usual Newtonian dynamics at 'large' accelerations and modified dynamics at 'small' accelerations (large and small here are relative to the Milgrom characteristic acceleration $a_0 \approx 10^{-10}$ ms^{-2}). In doing so, the observed flattening of galaxy rotation curves and the empirical Tully-Fisher relation can both be satisfied.

Recent work [2–4] on an experimental observation of MONDian like theories suggests that anomalously large tidal stresses should be present around gravitational saddle points. The proposed LISA Pathfinder (LPF) mission [5] should be capable of probing acceleration scales upto a_0, this is of special importance because here we would be probing the onset of MONDian behaviour, rather than the acceleration scales dominated by MOND. The purpose of this short paper is extend previous analysis done of the MOND saddle point science case, based upon a scenario where a mission extension is granted. The extension would involve redirecting the spacecraft from Lagrange point L1 to a saddle of the Earth-Moon-Sun system [6] once its nominal mission at L1 is completed. Previous analysis have only considered only non relativistic limits resulting from theories such as TeVeS or ones which are AQUAL-like, here we will examine how these predictions vary using instead a different formulation of MOND.

The structure of this paper is as follows, first we review the non-relativistic limits of the various modified gravity theories in the literature. In section III we solve the Poisson equation for this theory numerically around the Earth-Sun saddle point. We consider typical signal to noise ratios for detecting MONDian effects with LISA Pathfinder in section V, our main results being summarised in figures 3 and 4. Some more general concerns regarding these type II theories are considered in the last section and we will conclude with some thoughts on using all of these results constraining modified gravity theories. We also present some analytical solutions and detail our numerical methods in appendices.

II. A REVIEW OF MONDIAN THEORIES

The literature contains a whole plethora of relativistic MONDian theories, but we find their relative differences (and somewhat complexity) arise from the requirement that they explain relativistic phenomena (such as lensing and structure formation) without resorting to dark matter. In the non-relativistic regime however, all of them reduce to just to only 3 types of non-relativistic limits, which we will label type I, II and III.

- **Type I** In these theories, the non-relativistic dynamics results from the joint action of the usual Newtonian potential $\nabla\Phi_N$ (derived from the metric via $g_{00} \approx -(1 + 2\nabla\Phi_N)$) and a fifth force field, ϕ, responsible for MONDian effects. The total potential acting on non-relativistic particles is their sum:

$$\Phi = \Phi_N + \phi \tag{1}$$

Whilst the Newtonian potential satisfies the usual Poisson equation:

$$\nabla^2\Phi_N = 4\pi G\rho \tag{2}$$

the field ϕ is ruled by a non-linear Poisson equation:

$$\nabla \cdot (\mu(z)\nabla\phi) = kG\rho \tag{3}$$

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where, for convenience, we pick the argument of the free function function μ as:

$$z = \frac{k}{4\pi}\frac{|\nabla\phi|}{a_0} \quad (4)$$

where k is a dimensionless constant and a_0 is the usual MOND acceleration. In general, we require that $\mu \to 1$ for $z \gg 1$ and $\mu \sim z$ for $z \ll 1$.

- **Type II** In these theories we also have $\Phi = \Phi_N + \phi$, but now the field ϕ is ruled by a driven linear Poisson equation, whose source depends on the Newtonian potential. In order to facilitate comparison with Type I theories, we write the equation for ϕ in these theories as:

$$\nabla^2\phi = \frac{k}{4\pi}\nabla\cdot(\nu(v)\nabla\Phi_N) \quad (5)$$

where the argument of free function ν is given by

$$v = \left(\frac{k}{4\pi}\right)^2\frac{|\nabla\Phi_N|}{a_0} \quad (6)$$

and we require that $\nu \to 1$ when $v \gg 1$ and $\nu \sim 1/\sqrt{v}$ for $v \ll 1$. It is the analysis of the anomalous tidal stresses around the Earth-Sun saddle in this non-relativistic limit that is the subject of this paper.

- **Type III** This was the original non-relativistic MONDian proposal, derived from a non-relativistic action principle (the so-called AQUAL [7]). Crucially, here non-relativistic particles are sensitive to a single field Φ which satisfies a non-linear Poisson equation:

$$\nabla\cdot(\tilde{\mu}(x)\nabla\Phi) = 4\pi G\rho \quad (7)$$

Again, μ is a free function with a suitably chosen argument:

$$x = \frac{|\nabla\Phi|}{a_0} \quad (8)$$

so that $\tilde{\mu} \to 1$ for $x \gg 1$ and $\tilde{\mu} \sim x$ for $x \ll 1$.

Virtually all relativistic MONDian theories proposed in the literature fall into these categories. Both TeVeS, Bekenstein's relativistic MONDian theory [8] and Sanders stratified theory [9] have a type I limit. Milgrom's bimetric theory [10, 11] can have either a type I or type II limit, whilst Einstein-Aether theories [12, 13] appear to be to the only theories with a non-relativistic limit of type III.

We stress that it is important that we consider formulations wedded to fully relativistic theories (rather than just a scalar Lagrangian theory like AQUAL), even though we are not directly testing these full theories themselves. Their cosmologies however, do provide the necessary constraints on the gravitational constant G, the *bare* value of G appears in the friedman equations and so from BBN constraints can be fixed to corrections of order ~ 0.01 or smaller. As we will see, this issue is not a trivial one in modified gravity theories. Deriving the full field equations and taking the weak field limit as necessary allows us to find the modified Poisson equations for these theories consistently. A more thorough investigation of the weak field limit in these theoreis is left for future work [14]

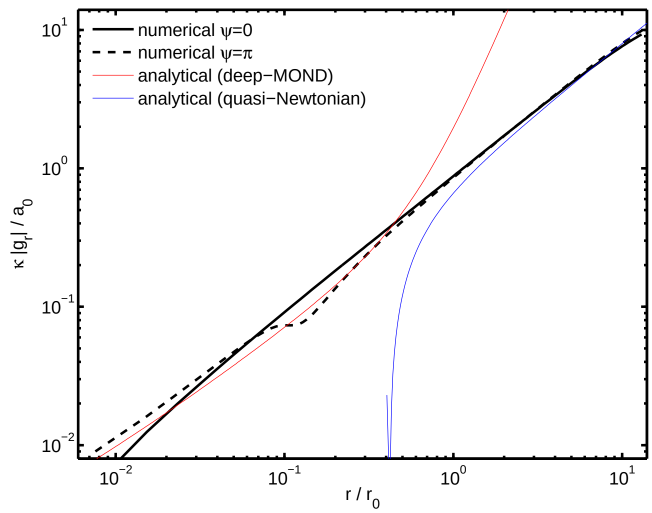

FIG. 1: A comparison between the numerical and analytical results for components of $\mathbf{g} = -\nabla\phi$ in the two-body Earth Sun case.

It has been argued that in some relativistic formulations of type II theories the bare G, that appearing in cosmology and the total G ruling the non-relativistic equation are the same. We call such theories type IIA, and for them $\nu \to 0$ in the Newtonian limit. Otherwise we call then type IIB, with a G renormalization, and with $\nu \to 1$.

III. NUMERICAL SOLUTIONS

To make further progress with the saddle point problem, we must move to using numerical solutions. We use a modification of the code used by Bevis et al (see Appendix B for details), which solved the modified Poisson equation over a lattice around the saddle. Non-uniform coordinates are employed to aid resolution in the deep MONDian regime. Due to the linear nature of equation (5), a variety of computation techniques can be used (notably FFT). Here however we have the advantage of using our code, whos solar system credentials have been examined previously [3]. Solving the sourced Poisson equation on the lattice to find $\delta\mathbf{F} = -\nabla\phi$, we find similar results to the Type I theories. We picked a ν function of the

form

$$\nu = \left(1 + \frac{1}{v^2}\right)^{1/4} \tag{9}$$

since it has similar properties to the μ function picked for type I theories, however it is simple to incorporate other functions in our code.

As before, we considered the Earth-Sun saddle (although three body extensions can be easily considered too), using a 257^3 lattice of physical size 10000 km with central resolution ≈ 2.6 km. As figure 1 shows, these produce a good match to analytical solutions (within their respective domains) and provide the appropriate interpolation in the intermediate region around the bubble boundary.

IV. TIDAL STRESSES AND SIGNAL TO NOISE RATIOS

Whilst predictions of MONDian forces are useful, LPF is sensitive to the tidal stresses. In addition to the issues regarding close flybys for the spacecraft (see [6] for a further account), two additional ones need to be taken into consideration here. One is that only transverse tidal stresses can be measured, due to practical issues regarding the spacecraft. The other is that ϕ field produces both a MONDian effective field and a rescaled Newtonian component, which both must be taken into account when calculating the tidal stresses, so we use the definition:

$$S_{ij} = -\frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{k}{4\pi}\frac{\partial^2 \Phi^N}{\partial x_i \partial x_j} \tag{10}$$

As stressed before, it is important that we know both field components to the same degree of accuracy or systematic errors can crop up due to the imperfect subtraction of the Newtonian field (see [4], section IV C).

We then move on to consider typical signal to noise ratios (SNR) produced by LPF in this formulation of MOND, using the techniques of noise matched filtering from gravitational wave searches [15]. The basic idea laid (as laid out in [4]) is to correlate a time series $x(t)$ with an optimized template designed to provide maximal SNR, given the signal shape $h(t)$ and the noise properties of the instrument. Whilst the specifics of the instrument noise won't be known properly until the satellite is *in situ*, there exist nominal requirements that it must meet, as well as best estimates for the noise signal waveform.

For the setup we described here, we have $h(t) = S_{yy}(vt, b, 0)$, where v is the velocity of the spacecraft (taken to be 1.5 km^{-1} throughout this work) and $t = 0$ corresponds to the point of closest saddle approach. In a more general setup, for an approximately constant velocity $\mathbf{v}$, a closest approach vector $\mathbf{b}$, and with the masses aligned along unit vector $\mathbf{n}$, we have

$$h(t) = n^i n^j S_{ij}(\mathbf{b} + \mathbf{v}t) \tag{11}$$

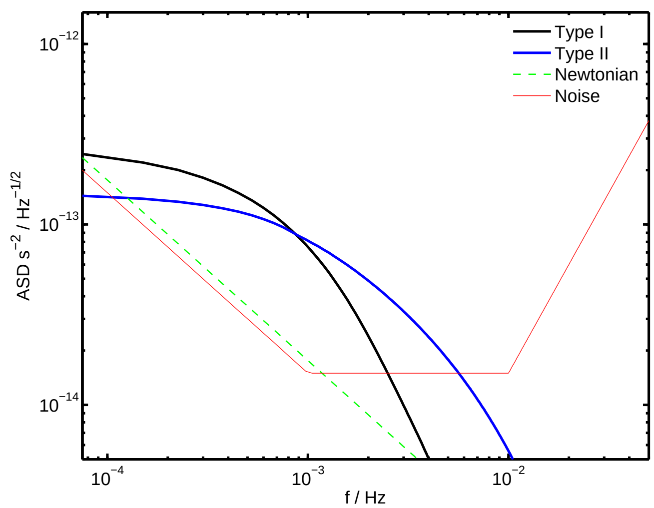

FIG. 2: ASD plot of the MONDian and Newtonian signals (multiplied by $\kappa/4\pi$), as compared to the basic noise ASD, for both the Type I and Type II theories, for a $y = 50$km fly-by, with $v = 1.5$ ms^{-1}. Using this data, we find the SNR is amplified from 28 to 35.

The maximal SNR, realized by the optimal template, is found to be:

$$\rho_{\rm opt} = 2\left[\int_0^\infty df \frac{\left|\tilde{h}(f)\right|^2}{S_h(f)}\right]^{1/2} \tag{12}$$

where $\tilde{h}(f)$ is the fourier transformed signal waveform, computed from the tidal stresses and $S_h(f)$ is the noise waveform. We plot these in figure 2 (in terms of amplitude spectral density (ASD), ie square root of power spectrum) to demonstrate that there is ample signal compared to the noise for LPF to detect MONDian behavior for a reasonable fly by. The headline figure is for a $b = 50$km flyby, with baseline noise 1.5x10^{-13} s$^{-2}/\sqrt{\rm Hz}$, the SNR observed should be ≈ 35, a distrinct improvement on the figure of ≈ 28 using the Type I theories.

As before, we can produce contours for typical SNRs produced at various impact parameters and baseline noises, see figure 3. In doing so, we suggest that larger SNRs (in comparison to type I theories) will in general result, for b's in excess of 1000km, as illustrated in figure 4. The most striking thing to take away from this plot is that even as far out as 1000km, there is still a marked difference between SNRs of the two theories.

V. FEATURES OF TYPE II THEORIES

Given the amplified MONDian signal that we showed was present in these theories, it is tempting to suggest that a null result would represent an even tighter constraint on these theories than compared to type I. We

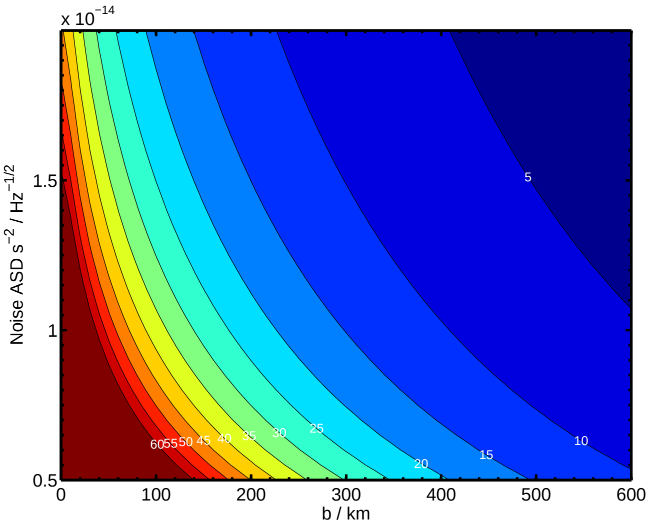

FIG. 3: Signal to noise ratio (SNR) contours for various impact parameters and baseline ASD noise. We set the $v = 1.5\ \mathrm{kms}^{-1}$. Calamitous assumptions would still lead to SNR in excess of 5. More optimistic ones (b around 50km or less, noise half way up the scale) would lead to SNRs easily around 55.

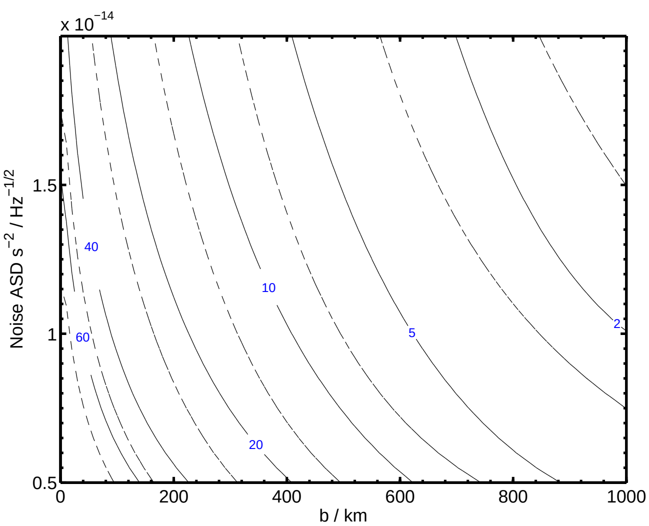

FIG. 4: Signal to noise ratio (SNR) contour lines for various impact parameters and baseline ASD noise waveforms comparing between type I and II theories. The solid lines are the typical SNR to be obtained in the type II theories and the dashed lines to their immediate left the corresponding type I line. We show for impact parameters up to $b = 1000$ km to demonstrate type II always substantially beats type I on SNR.

should however be clear to distinguish between a *stronger* MONDian signal and the relative size of the Deep MONDian region r_0. In these theories, with 'simple' single power law ν functions, the bubble size r_0 remains unchanged, this is obvious (eg from analytical solutions, see appendix A). The tidal stresses however are amplified at larger frequencies (see figure 2), the integrated effect of which is a larger SNR. This however is not the same thing as a theory with a larger MOND bubble, which would boost SNRs at *all frequencies* (equivalent to making a closer flyby).

A. One or Two Field Models

We observe that (5) can be rewritten as a manifestly one field model in terms of the gravitational potential Φ and an (auxilliary) Newtonian Potential Φ_N as

$$\nabla^2\Phi = \nabla\cdot(\hat{\nu}(w)\nabla\Phi_N) \tag{13}$$

where

$$\hat{\nu}(w) = 1 + \frac{k}{4\pi}\nu(v) \tag{14}$$

and $w = |\nabla\Phi_N|/a_0$. An observation we can make in this case is, that for large F_N, $\nu \to 1$, making

$$G_{Ren} = \left(1 + \frac{k}{4\pi}\right)G \tag{15}$$

whereas if we picked a manifestly one field model (IIA), there would be no G renormalisation and we would have $\nu \to 0$ in the large F_N limit. This has the entertaining consequence of being equivalent to a model with $\mu \to \infty$, a result we discuss in the next section.

B. Divergent functions - a discussion

In Type I theories, we have the case where

$$\nabla\cdot(\mu\,\mathbf{F}_\phi) = \frac{k}{4\pi}\nabla\cdot\mathbf{F}_N \tag{16}$$

which *under the assumption of spherical symmetry,* we can ignore curl terms and reduce to

$$\mu\,\mathbf{F}_\phi = \frac{k}{4\pi}\mathbf{F}_N \tag{17}$$

We find there are broadly two classes of μ functions, those which converge to some constant and those which diverge in the large acceleration limit.

Lets consider the sub class of μ's parameterised by α, with $\alpha = 1$ first suggested in [16] as being a good fit to galactic data:

$$\mu = \frac{z}{1 - \frac{\alpha z}{k'}} \tag{18}$$

where for brevity $k' = k/4\pi$. Here it appears that for $z \to \infty$, $\mu \to -\frac{k'}{\alpha}$, however it turns out if we investigate the behaviour of z properly, by substituting into (17)

$$\frac{z^2}{1 - \frac{\alpha z}{k'}} = v \tag{19}$$

and then solving for z (picking solutions for $z \geq 0$)

$$z = \frac{\alpha v}{2k'} \left[-1 + \sqrt{1 + \frac{4(k')^2}{\alpha^2 v}} \right] \tag{20}$$

and so as $F_N \to \infty$

$$\mathbf{F}_\phi = \frac{a_0}{\alpha} \frac{\mathbf{F}_N}{F_N} \tag{21}$$

i.e. z goes to a constant as μ diverges and we never reach the other branch of solutions for $\mu(z > \frac{k'}{\alpha})$. This also means that since the MONDian field approaches $\approx a_0$ asymptotically, we are always in the region where 'deep' MOND effects are present (see section VIA of [4]). We can however use (20) to write this as a ν function

$$\nu = \frac{\alpha}{2k'} \left[-1 + \sqrt{1 + \frac{4(k')^2}{\alpha^2 v}} \right] \tag{22}$$

which evidently has $\nu \to 0^+$ in the large F_N limit. This has the predictable conclusion that with these functions, $G_{Ren} = G$. It is however exactly these functions which are required in a one field model.

We can schematically write a function with the same behaviour as

$$\nu = \frac{\sqrt{v}^{-1}}{1 + \frac{\alpha}{k'}\sqrt{v}} \tag{23}$$

ie the usual $\nu \to 1/\sqrt{v}$ in the MONDian regime but moving to a different power law for larger accelerations $\nu \to 1/v$. This notion of switching power laws at different acceleration scales is the subject of the next section.

C. Designer ν functions

One requirement of an experimental test of MONDian theories is to either conclusively show there is an extra gravitational force present in the low acceleration regime or in the case of a null result, provide clear constraints for theories. It is reasonable now to suggest that flyby impact parameters that can probe the MOND bubble (ie $r \leq 380$km) will be possible. With this in mind, lets consider probing that regime and the effect of seeing no signal above the Newtonian background and the noise. At best one, can then posit SNR of order 1 and the size of the bubble is smaller than originally thought. We would then have to design a transition function as such to escape the net of LPFs capabilities in this way. We considered this approach with the μ functions of type I (see section VI of [4]) and here we apply it similarly to the ν functions of type II.

We start by fixing the asymptotica of our problem, which are $\nu \approx 1/\sqrt{v}$ in the astrophysical regime (ie $F_N \leq a_0$) and $\nu \approx 1$ in the quasi-Newtonian regime as such to be washed out far away from the saddle by superior size of the Newtonian potential. In the intermediate regime (which we are probing with LPF), we can suggest a different power law, as such making this a two scale model, with the power n and a_0 now the two independent features of these theories to be constrained.

$$\nu \approx 1/\sqrt{v} \quad \text{for} \quad v < \left(\frac{k}{4\pi}\right)^2 \tag{24}$$

$$\nu \approx \left(\frac{v^{trig}}{v}\right)^n \quad \text{for} \quad \left(\frac{k}{4\pi}\right)^2 < v < v^{trig} \tag{25}$$

$$\nu \approx 1 \quad \text{for} \quad v > v^{trig} \tag{26}$$

where the point when non-Newtonian behaviour in ϕ is triggered can be interchangeably pinpointed by:

$$v^{trig} = \left(\frac{\kappa}{4\pi}\right)^{1-\frac{1}{n}} \tag{27}$$

$$a_\phi^{trig} = a_0 \left(\frac{\kappa}{4\pi}\right)^{-\frac{1}{n}} \tag{28}$$

$$a_N^{trig} = a_0 \left(\frac{\kappa}{4\pi}\right)^{-1-\frac{1}{n}} \tag{29}$$

We still have that when $a_N < a_0$ the field ϕ dominates Φ_N as per our requirements, but now the intermediate region, where ϕ hasn't yet dominated but is already non-Newtonian, is in a narrower band of accelerations $a_0 < a_N < a_N^{trig}$. As a result, the MOND bubble shrinks in this model according to [1]

$$r_0 \approx 380 \left(\frac{\kappa}{4\pi}\right)^{\frac{n-1}{n}} \text{ km} \tag{30}$$

Making a model independent statement on which power would be required for a SNR of order 1 remains beyond our reach, simply because a null result can only lead us to conclude we are probing the region $b \gg r_0(n)$. Equivalently this statement means we are probing the transient from truely (albeit rescaled) Newtonian behaviour $\nu \sim 1$ and power law behaviour $\nu \approx v^n$. This however can vary wildly even between 'simple' ν functions of one power.

We can however perform an order of magnitude argument, assuming a designer function, say:

$$\nu(v) = 1 + \left(\frac{v^{trig}}{v}\right)^n \tag{31}$$

[1] Where again we notice that the smallest the bubble can shrink to in this model is $r_0 \approx 1$km.

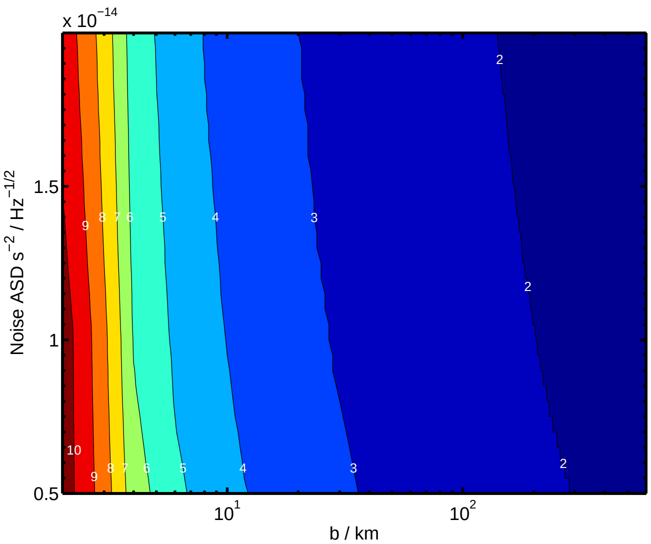

FIG. 5: Contours of the power n needed to obtain SNR $= 1$ for different impact parameters and noise levels, upto $b = 600$ km For $n > 2$, we find our functions become increasingly contrived in order to explain a null result. This suggests that a negative result at $r_0/2$ or closer would need to be explained away differently.

We can write

$$\mathbf{F}_\phi = \frac{k}{4\pi}\,\nu\,\mathbf{F}_N \tag{32}$$

since it is reasonable to ignore divergence free sources here ($\nu \approx 1$ at these scales). Breaking this up into

$$\mathbf{F}_\phi = {}^0\mathbf{F}_\phi + \delta\mathbf{F}_\phi \tag{33}$$

with rescaled Newtonian component ${}^0\mathbf{F}_\phi = \frac{\kappa}{4\pi}\mathbf{F}_N$. Substituting in and solving gives:

$$\delta\mathbf{F}_\phi = \frac{k}{4\pi}\,(\nu - 1)\,\mathbf{F}_N = \left(\frac{4\pi}{k}\frac{a_0}{|\mathbf{F}_N|}\right)^n \mathbf{F}_N \tag{34}$$

from which the tidal stresses can be inferred. We notice that in this instance, this equation is exact since $\nu = \nu(F_N)$, whereas for type I theories, this process was done perturbatively.

The results of using this model are presented in figure 5 (kindly borrowed from [4]), indicating the value of n needed for a given b and noise level in order for a SNR of one to be obtained (and so a negative result be acceptable). Once we start probing deep into the MOND bubble, a rather 'fine tuned' designer ν becomes necessary to accommodate a negative result. The naive expectation that these null results are more constraining on type II theories is this a misnomer, because we are not probing the deep MONDian regime here, merely the transient from quasi-Newtonian behaviour. In this way, a null result would make differentiating between theories inconclusive.

VI. CONCLUSIONS

In this paper we solved the non-relativistic limit of a different MONDian theory formulation (here classed Type II) for the saddle points in the two-body Earth-Sun system. We probed the additional MONDian effects and forecasted for LPF flyby probable detection possibilities. We suggest that the chances of detection of a MONDian signal are enhanced compared to Type I theories (eg those resulting from TeVeS), with typical SNRs for a 50km flyby rising from 28 to 35. Another key result is that the boosted SNRs remain higher even as far out as $b \approx 1000$km or more, enhancing observational prospects even for catastrophic impact parameters. We also suggest that in the event no anomalous tidal stresses are observed, the experiment can put constraints on these theories, however they remain no better than in Type I, due to the nature of the regime we would be probing. Further work however remains to be done into ways to differentiate between MONDian formulations. One significant difference here is that there are no curl type forces present, due to the linear nature of the governing equation. Whilst a null result would place constraints on these theories, a positive one would do little to guide us towards a particular formulation or even form μ or ν function. We are currently working on techniques to constrain or perhaps extract some features of the function from potential data obtained.

Acknowledgments

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Appendix A: Analytical Solutions

In this work we will follow the conventions and notations used previously. Our experimental observations

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[2] During the preparation of this work, a separate group has submitted similar work on the arXiv [17] dealing with solving QMOND in the Solar System. Whilst their work involves solving a similar sourced Poisson equation and they report similar results to ours, ours is a two field model. They on the other hand use a one field model, where the MONDian field is an auxiliary one and the observable is the gravitational potential Φ. Our techniques are kept similar to previous work to allow more direct comparison between the two theories. The reason that the two approaches are so complementary for this test, is simply that we are free of sources at the saddle, so the resulting equations are the same, this result would not be in general true.

will be centred around the saddle point so it makes sense that our coordinate system is too. For solving the Poisson equation about the saddle, we pick a radial system of coordinates, however when we consider spacecraft trajectories we revert to Cartesian coordinates.

Although we have freedom in picking the form of the ν function, the one used throughout this work was

$$\nu = \left(1 + \frac{1}{v^2}\right)^{1/4} \tag{A1}$$

This was picked as so to easily compare with previous results from type I theories.

About the saddle point, we have the Newtonian field

$$-\nabla\Phi_N = \mathbf{g}_N = Ar\mathbf{N} \tag{A2}$$

where

$$\begin{aligned} \mathbf{N}(\psi) &\equiv N_r\mathbf{e}_r + N_\psi\mathbf{e}_\psi & \text{(A3)}\\ N_r &= \frac{1}{4}[1 + 3\cos(2\psi)] & \text{(A4)}\\ N_\psi &= -\frac{3}{4}\sin(2\psi). & \text{(A5)} \end{aligned}$$

and hence

$$v = \frac{r}{r_0}N \tag{A6}$$

where $N = |\mathbf{N}| \sim \mathcal{O}(1)$.

Replacing the Newtonian force in (5) with (A2), using the fact that at the saddle $\nabla^2\Phi_N = 0$ and expanding out gives our expression in this linear regime:

$$\nabla^2\phi = \frac{a_0}{2}\left(\frac{4\pi}{k}\right)\left(\frac{1}{r_0\, r}\right)^{1/2}\left(\frac{r_0^2}{r_0^2 + (rN)^2}\right)^{1/4} \left(\frac{N_r}{N^{1/2}} + \frac{N_\psi}{N^{3/2}}\frac{\partial N}{\partial\psi}\right) \tag{A7}$$

The problem here is akin to electrostatics, where we must solve the sourced Poisson equation subject to the appropriate boundary conditions, which here are that δF_ψ vanishes (and δF_r equate) at $\psi = 0$ and π, such that we avoid a jump in the field at $\psi = \pi/2$ [2].

1. 'Deep MONDian'

For $r \ll r_0$, it's clear (A7) reduces to:

$$\nabla^2\phi = \frac{a_0}{2}\left(\frac{4\pi}{k}\right)\left(\frac{1}{r\, r_0}\right)^{1/2}\left(\frac{N_r}{N^{1/2}} + \frac{N_\psi}{N^{3/2}}\frac{\partial N}{\partial\psi}\right) \left\{1 - \frac{3}{4}v^2 + ...\right\} \tag{A8}$$

where the angular parts of the leading order term neatly reduce to

$$\frac{7 + 9\cos 2\psi}{(2(5 + 3\cos 2\psi))^{5/4}} = g(\psi) \tag{A9}$$

and the higher powers are supressed since $v \ll 1$.

We use an Ansatz for the MONDian field

$$\phi = C_1\, r^a\, F(\psi) \tag{A10}$$

where C_1 is some constant to be fixed from the source term above. This gives rise to a sourced 2nd order ODE:

$$r^{a-2}\left(a(a+1)F + \cot(\psi)\, F' + F''\right) = r^{-1/2}\, g(\psi) \tag{A11}$$

where $' = \partial/\partial\psi$ and

$$C_1 = \frac{4\pi}{k}\frac{a_0}{\sqrt{r_0}} \tag{A12}$$

We find that the solutions of the homogenous (unsourced) equation are Legendre Polynomials of order a, with the form of (A11) suggesting $a = 3/2$. The full inhomogeous (sourced) solution is then given by the variation of parameters method and the solutions given as a Fourier series expansion in even components of ψ:

$$\begin{aligned} F(\psi) \approx &- 0.0236032 - 0.188606\cos(2\psi) & \text{(A13)}\\ &+ 0.010834267\cos(4\psi) + ... \end{aligned}$$

In addition, at the saddle we have region where $|\mathbf{g}_N| = 0$ and so we need to consider solutions to the Laplace equation

$$\nabla^2\phi = 0 \tag{A14}$$

which subject to smoothness and continuity conditions being satisfied and regularity at the origin, can be written in general by

$$\phi_L = a_0\left(\frac{4\pi}{k}\right)\sum_\ell A_\ell\, r^\ell\, P_{2\ell}(\cos\psi) \tag{A15}$$

where $P_{2\ell}(\cos\psi)$ are Legendre polynomials,

$$A_\ell = \frac{a_\ell}{r_0^{\ell-1}} \tag{A16}$$

and a_ℓ are dimensionless constants to be found by matching solutions at the intermediate MONDian regime. Our normalisation is picked to be of the same form as the sourced solutions, so that $\nabla\phi$ has units of acceleration. It is important to consider that we need only expand a few terms out form the Laplace contribution when comparing against data, since the region of validity of these solutions is small.

2. 'Quasi-Newtonian'

For $r \gg r_0$, it's clear that (A7) reduces to:

$$\nabla^2\phi = a_0\left(\frac{4\pi}{k}\right)\frac{r_0}{r^2}\frac{1}{2N^2}\left(N_r + \frac{\partial N}{\partial\psi}\frac{N_\psi}{N}\right)\left\{1 - \frac{3}{4}\frac{1}{v^2} + ...\right\} \tag{A17}$$

where the leading order angular term simplifies to become

$$\frac{2(7+9\cos 2\psi)}{(5+3\cos 2\psi)^2} = h(\psi) \tag{A18}$$

and again higher powers are suppressed since $v \gg 1$.

Our Ansatz for the MONDian field here is

$$\phi_1 = C_1\, H_2(\psi) + C_2 \ln\left(\frac{r}{r_0}\right) \tag{A19}$$

since if used a similar ansataz as before, we wouldn't be able to satisfy smoothness and continuity on $[0,\pi]$ (we can however use them for higher order terms in the expansion of ν). Computing the Laplacian gives

$$\nabla^2\phi = \frac{C_1}{r^2}\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sin H_2') + \frac{C_2}{r^2}r_0 = a_0\left(\frac{4\pi}{k}\right)\frac{r_0}{r^2}h(\psi) \tag{A20}$$

allowing us to set

$$C_1 = C_2\, r_0 = \left(\frac{4\pi}{k}\right) a_0\, r_0 \tag{A21}$$

Integrating out once then gives

$$\sin\psi\,\frac{\partial H_2}{\partial\psi} = \int (h-1)\sin\psi\, d\psi + A \tag{A22}$$

Again requiring that we satisfy our boundary conditions means we find

$$A = -\left(\frac{3}{2} + \frac{\pi}{3\sqrt{3}}\right) \tag{A23}$$

We can then solve (A22) for H_2 and expand as a fourier series in ψ

$$\begin{aligned} H(\psi) \approx &- 0.22921 + 0.287573\cos 2\psi \\ &- 0.116339\cos 4\psi + ... \end{aligned} \tag{A24}$$

Further terms in the expansion of (A17) result in a series expansion for the MONDian field ϕ

$$\phi = \phi_2 + \frac{4\pi}{k}a_0\sum_{n=2}^{\infty} C_n\left(\frac{r}{r_0}\right)^{2-2n} H_n(\psi) \tag{A25}$$

where $H_n(\psi)$ satisfies the sourced ODE

$$n(n+1)H_n + \cot(\psi)H_n' + H_n'' = h_n \tag{A26}$$

and h_n is given by

$$h_n(\psi) = \frac{2^{3n/2-2}(7+9\cos 2\psi)}{(5+3\cos 2\psi)^{n/2+1}} \tag{A27}$$

In addition to these solutions, we also always have the background rescaled Newtonian contribution

$$\Phi_N = \frac{a_0}{8}\left(\frac{4\pi}{k}\right)\frac{r^2}{r_0}(1+3\cos 2\psi) \tag{A28}$$

In the limit $r\to\infty$, we recover this rescaled Newtonian limit. [3]

Appendix B: Details of Numerical Method

Here we describe our numerical algorithm. We consider an adaptation of the the numerical set up used previously [3], representing $\mathbf{g} = -\nabla\phi$ on the sites of a non-uniform lattice around the saddle point. We frame the system as a pair of vector equations

$$\nabla\cdot\mathbf{g} \;=\; \frac{k}{4\pi}\nabla\cdot\left[\nu(g^N)\mathbf{g}^N\right] \tag{B1}$$

$$\nabla\times\mathbf{g} \;=\; 0 \tag{B2}$$

Our initial conditions are at each site set as

$$\mathbf{g}_0 = \frac{k}{4\pi}\mathbf{g}^N \tag{B3}$$

we then compute the discrete divergence

$$D_{\mathbf{x}} = \sum_j \frac{g^j_{\mathbf{x}} - g^j_{\mathbf{x}-\mathbf{j}}}{\Delta^j_-} = D^N_{\mathbf{x}} \tag{B4}$$

where we will use the compact notation

$$\Delta^j_- = r^j_{\mathbf{x}} - r^j_{\mathbf{x}-\mathbf{j}} \tag{B5}$$

$$\Delta^j_+ = r^j_{\mathbf{x}+\mathbf{j}} - r^j_{\mathbf{x}} \tag{B6}$$

and have discrete source term

$$D^N_{\mathbf{x}} = \sum_j \frac{\nu_{\mathbf{x}}(g^j_{\mathbf{x}})^N - \nu_{\mathbf{x}-\mathbf{j}}(g^j_{\mathbf{x}-\mathbf{j}})^N}{\Delta^j_-} \tag{B7}$$

such that at each site we locally solve (B1) whilst ensuring (B2) is satisfied globally. As before, we can consider changes in the field $g^j \to g^j + \delta g^j$ that keep the discrete curl condition (B2) satisfied, where

$$\delta g^j_{\mathbf{x}} = +\frac{C_{\mathbf{x}}}{\Delta^j_+} \tag{B8}$$

$$\delta g^j_{\mathbf{x}-\mathbf{j}} = -\frac{C_{\mathbf{x}}}{\Delta^j_-} \tag{B9}$$

These changes are then reflected when we compute the change in the discrete divergence $\delta D_{\mathbf{x}}$

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[3] The choice of background terms is not unique, but rather dependent on the form of ν. If we had picked a ν with both odd and even powers in its expansion (i.e. $\nu \approx 1 + \alpha_1/v + ...$), there would also be a background ϕ_1 contribution since $\nabla\phi_1$ would be an angular function only. However this terms contribution in the $r\to\infty$ limit is negligible.

Ensuring that the change in the divergence at each step now gives

$$D_{\mathbf{x}} + \delta D_{\mathbf{x}} = D_{\mathbf{x}}^N \tag{B10}$$

meaning that the change in the MONDian field at each site is now given by

$$\delta g_{\mathbf{x}}^j = -\frac{D_{\mathbf{x}} - D_{\mathbf{x}}^N}{\delta D_{\mathbf{x}}} \frac{C_{\mathbf{x}}}{\Delta_+^j} \tag{B11}$$

such that as we cycle through the lattice and the $\mathbf{g}$ field converges, the additional changes to $g_{\mathbf{x}}^j$ lessen. We achieve faster convergence using a successive over relaxation method (SOR) as before, by scaling the field as

$$\delta g_{\mathbf{x}}^j \to \lambda \delta g_{\mathbf{x}}^j \tag{B12}$$

where λ is the over-relaxation parameter and is larger than unity. We begin with $\lambda = 1$ and increase it once the field is settling down, since high values of λ can initially result in the RMS value of $|\delta D_{\mathbf{x}}|$ increasing, whilst we are looking for $|\delta D_{\mathbf{x}}| \to 0$.

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